Change Ringing

Where Math and Music Meet

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Project Goals

Change ringing is the ringing of bells by a group of ringers in a prescribed pattern, so that a set of bells is rung in all possible sequences. Change ringing is thus not only a loud communal hobby, but also a mathematically and computationally interesting problem at the intersection of music and mathematics. In this project, I explored change ringing from the perspective of:

- **Music and History** –When and how did change ringing originate?
- **Mathematics** –How can change ringing be described mathematically?
- **Simulation** –Can I create a Python program to find valid change ringing sequences and mimic change ringing?

Figure 1. Change ringing on the bells at St. Mary Redcliffe, Bristol

Introduction to Change Ringing

The practice of change ringing originated in the 17th century, when church bells in England were first hung on full wheels (see figure 2) A bell on a full wheel can be rung deliberately and, with the help of the stay on the wheel, can pause in an upside-down position to wait for the next stroke. Using this technology, a group of ringers can ring a set of bells in a deliberate sequence.

To see what change ringing looks and sounds like, scan the QR code at the top of the poster (figure 1).

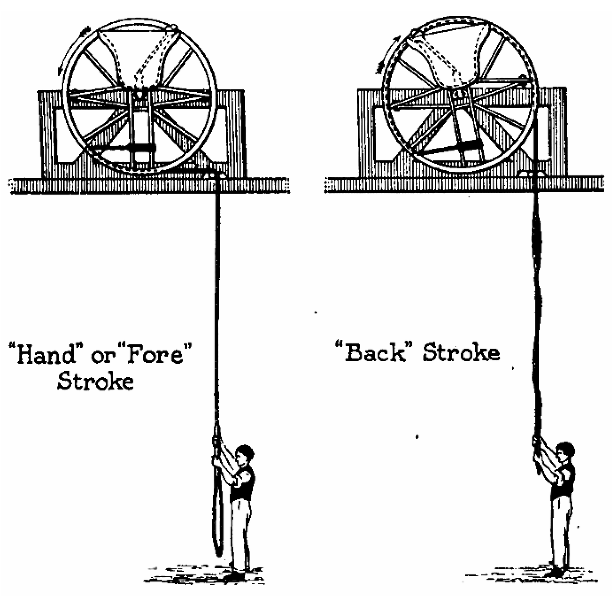

Figure 2. The two strokes that can be used to ring a bell on a full wheel

Some important change ringing vocabulary:

- A **change** is any sequence in which each bell is rung once.
- **Rounds** is a special change in which the bells are rung in sequence of pitch from the highest to the lowest.
- Ringers learn **methods** that connect changes in a systematic way. For example, *plain hunting* describes a bell whose position moves systematically up and down through a sequence of changes.
- An **extent** is a sequence in which all possible permutations of the bells are rung without repeating any of them except for rounds.

The main rules for ringing extents are:

- The first and last change are both rounds and no other change is repeated.
- A bell can only swap with the bells rung immediately before or after it .
- To keep the sequence interesting for the ringers, a bell should not stay in the same position in the sequence for more then two changes.

The bell towers used by change ringers typically have between 3 and 12 bells. The number of changes required to ring an extent rises rapidly with the number of bells: an extent of 3 bells requires only 7 changes, but an extent of 8 bells requires 40'321. So much time is required to ring an extent on over 7 bells that a full extent on 8 bells has only been rung once without changing the ringers during the extent. At a rate of 30 changes per minute, this means 22 hours and 24 minutes of continuous bell ringing!

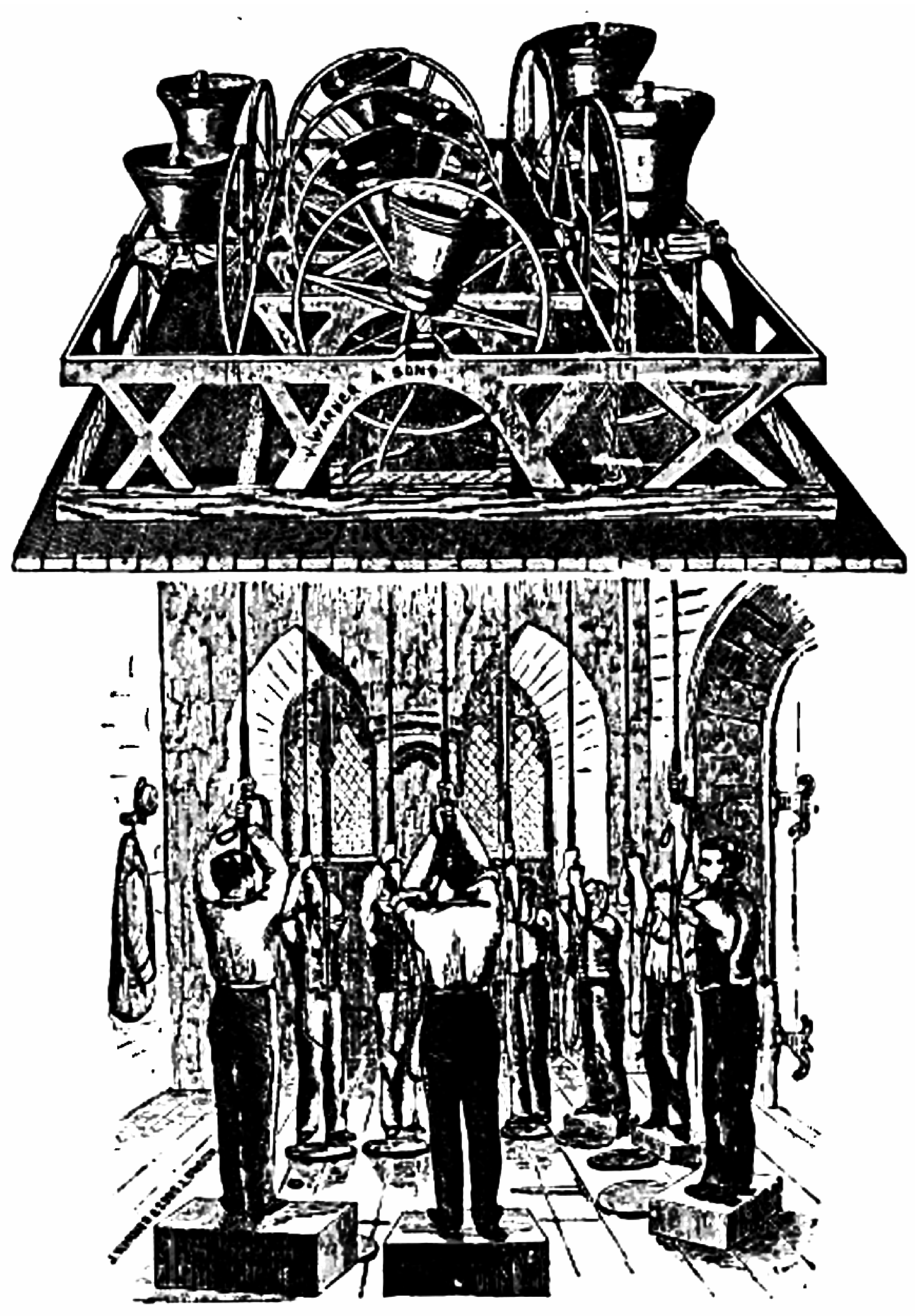
Figure 3. Change ringers

The Mathematics of Change Ringing

In change ringing we have a set of n bells, where $n \geq 2$. The bells in the set are ordered in order of descending pitch and numbered 1 to n. A change corresponds to a permutation of the ordered set [n] containing all integers from 1 to n in their natural order.

Group theory

A group is an algebraic structure $(G, *)$, in which there is a set G and a binary operation $*$ that satisfy the property of associativity and have an identity and and inverse element. Let Ω be a set with $n > 0$ elements; these elements can be rearranged to create **permutations** of Ω which corresponds to a change. The set of all permutations of Ω form a group with the operation being bijection. This group is called a **symmetric group**. A symmetric group has $n!$ elements. (An extent has $n! + 1$ elements, because rounds are repeated.)

For example, the symmetric group with 4 elements, or S_4, is generated by the set $\Omega = 1, 2, 3, 4$ and the group contains $4! = 24$ elements. One permutation of this group would be $(1, 2, 3, 4) \to (1, 3, 2, 4)$, the second and third elements are swapped. This swap can also be denoted in cycle notation as $\sigma = (23)$, where the two numbers correspond to the positions of the two elements being swapped.
For change ringing with 4 bells there are only four possible swaps: $A = (12)(34)$, $B = (23)$, $C = (34)$, $D = (12)$. This is due to the fact that only two bells rung after each other can switch places. One method on 4 bells is shown in figure 5. Note that in this method, bells sometimes remain in the same position for more than two changes, so technically this is not a valid extent.

Graph Theory

This group can be depicted by a Cayley graph. Each vertex of a graph corresponds to a permutation, and edges between the vertices show how elements are swapped between two permutations. Four edges connect to each vertex and together the vertices and the edges form a graph. To find possible change ringing extents, cycles have to be found in the graph starting at rounds and visiting each vertex before returning to rounds. Such a cycle is called a Hamiltonian cycle. The Cayley graph for the method in figure 5 is shown below in figure 4.

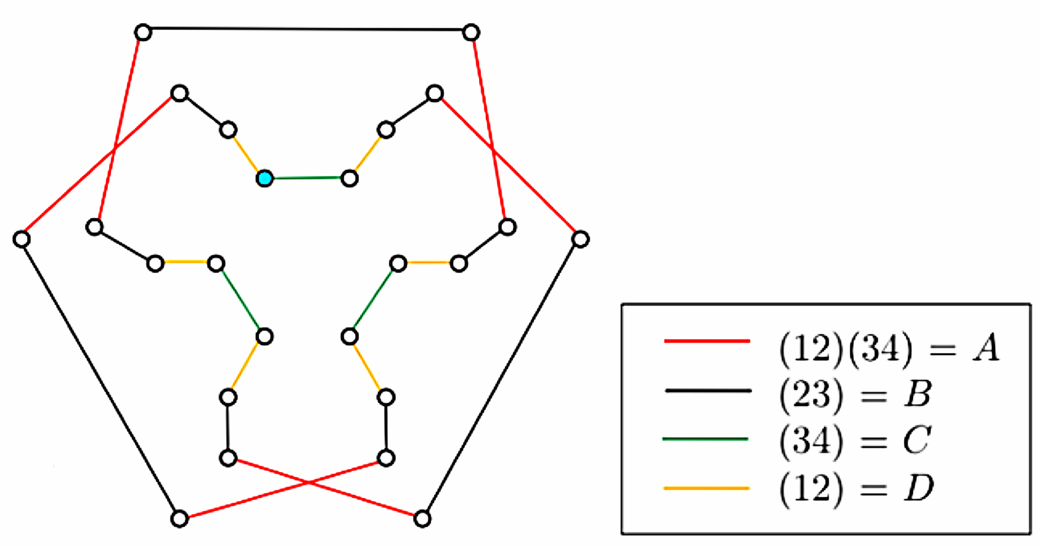

Figure 4. The Cayley graph for the method: $(DB(AB)^2DC)^3$ with the blue vertex representing rounds

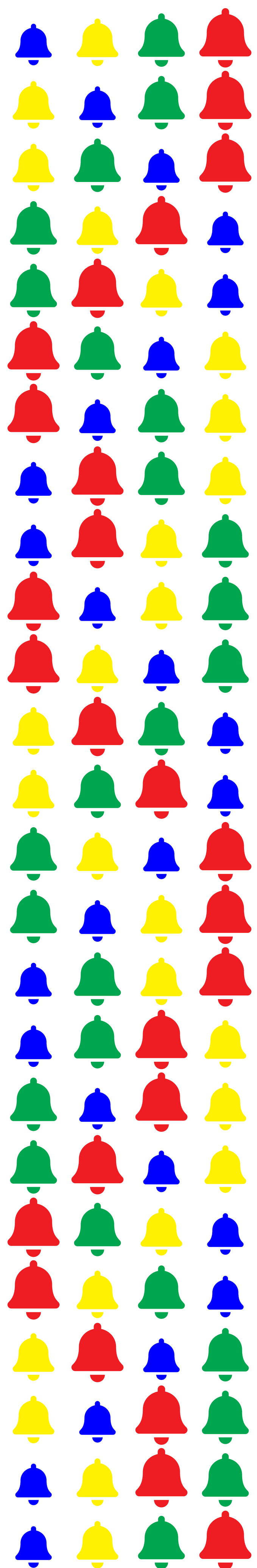
Figure 5. A method on 4 bells: $(DB(AB)^2DC)^3$

Simulation of Change Ringing

This project included creating a program in Python to find valid extents for a set of 4 bells using a recursive search method. Briefly:

The starting change was always rounds, the list [1, 2, 3, 4].

From the current change, the program makes one of the four allowed transitions (see math section) and added the resulting permutation to the working list. Then a series of checks is performed:

→ Has this change already been rung?

- If yes:

  → Is the change rounds, and does the working list have a length of 25?
  - If the answer is yes to both parts of the question, we have found a valid extent. ⇒ SUCCESS
  - If the duplicated change is not rounds or the length of the working list is not 25, this is not a valid extent. ⇒ STOP

- If no:

  → Have all the bells changed positions in the previous two changes?
  - If yes: The sequence will continue to be interesting for the ringers and this could be a valid extent ⇒ CONTINUE looking
  - If no: One of the ringers will become bored and this is not a valid extent. ⇒ STOP

This program found 24 valid extents for $n = 4$ bells. Interestingly, all 24 valid extents consist of three repeated sequences of transitions, and all of them are variations on the three known methods that produce valid extents:

$(AB)^3AC$, $ABAD(AB)^2$ and $ABADABAC$

This is because for each valid method, the method shifted by an arbitrary number of transitions is also valid.

Simulated Change Ringing

With the permission of the Kirchenkommission of Maschwanden ZH, I made recordings of their four church bells. One of the outputs of the recursive-search program is chosen at random and using the Python package pygame, these recordings are played in a sequence defined by the chosen output. The result for one such sequence can be heard by following the link in the QR code on the left (figure 6).

Figure 6. Demonstration of the program in action

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